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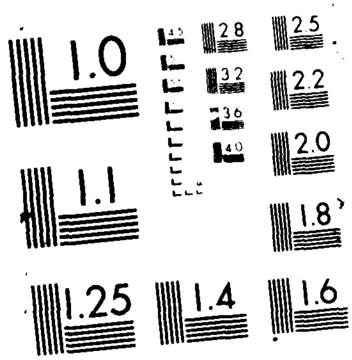
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Research Report CCS 582

EXPLICIT CONSTRUCTION AND ANALYSIS OF SOME EFFICIENT EMPIRICAL PRODUCTION FUNCTIONS IN DATA ENVELOPMENT ANALYSIS

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ABSTRACT

In contrast to classical econometric work which only tests data for consistency with a special class of production functions, new theory and explicit construction by Data Envelopment Analysis of the empirical Pareto-Koopmans efficient production function is developed for data sets which satisfy two conditions met in all previous real applications of DEA known to the authors. The construction requires no additional computation beyond that of the DEA tests.

KEY WORDS

Data Envelopment Analysis

Pareto-Koopmans efficiency

Multi-criteria Programming

Empirical Efficient Production Functions

1. Introduction

The "Foundations of Data Envelopment Analysis for Pareto-Koopmans Efficient Empirical Production Functions" by Charnes, Cooper, Golany, Seiford and Stutz (1985), hereinafter referred to as "Foundations," initiated a basic theoretical analysis and interpretation of Data Envelopment Analysis (DEA) as an approach for analyzing and evaluating the (relative) efficiency of performance of DMUs (Decision Making Units) by reference to production possibility sets specified directly instead of implicitly through point-to-set mappings required to satisfy properties imposed by a priori axioms as in Shephard (1970). Additionally, the DEA approach and orientation involved production possibility sets specified or estimated directly from empirical data which conform to real managerial possibilities and from which the efficient empirical production function is to be specified insofar as it can be known from the data and possibility set assumed in contrast to the emphasis and approaches associated with the abstract axiomatic econometric tradition.

In section 2, we bring forth the differences in key preoccupations of research in economic production theory with those of DEA by references to papers of Hanoch and Rothschild (1972), of Diewert and Parkan (1983) and, in still another direction by Färe, Grosskopf and Lovell (1985) and Debreu (1951).

In section 3 we develop basic concepts of and theorems for geometric elucidation and analysis of the empirical efficient production functions of DEA including a new lemma on optimal solutions to the general linear programming problem (for arbitrary ordered fields of scalars). Then, restricting ourselves to two conditions on empirical data which have held in every DEA application we have made, we show that the mathematical structure of the efficient empirical function is so simplified that it is available immediately in analytic form without further computational work on top of the

DEA tests. The efficient empirical production is then, moreover, piece-wise linear and continuous from piece to piece.

2. DEA and Classic Economic Approaches

While the orientation in DEA is toward construction of efficient production functions with best possible conformance to observed empirical data, approaches taken and objectives in classical approaches are quite different. This may be made more concrete by reference to a very early and a very recent article which are directed toward testing whether observed data conform to requirements specified in formal economic theories of production e.g. of the Shephard type. Thus, Hanoch and Rothschild (1972) is an early example of the latter approach to determine whether empirical observations have the desired properties via linear programming techniques. Diewert and Parkan (1983) provide a recent example in this same tradition. In both cases, the orientation is toward applying linear programming formulations to different bodies of data in order to ascertain whether they are globally consistent with the properties postulated for the production functions that are used in economic theory.

As noted in Diewert and Parkan, p. 131, for example, "it is important to know whether a failure to satisfy economic regularity conditions [in a statistically fitted function] is due to the use of an inappropriate functional form or whether the given data are simply inconsistent with any functional form satisfying the appropriate regularity conditions." In fact, in the latter vein, Hanoch and Rothschild, p. 273, report that a body of data used by others for purposes of statistical estimation is dangerous because their tests show that these data fail to satisfy the regularity conditions required by the economic theory of production.

In contrast, DEA as described in Foundations 1985 starts from observations on empirical data and proceeds to determine locally, with each DMU, a "facet" spanned by efficient DMUs from which a piece of the efficient empirical function (of a much wider class than those in this economic literature) together with output shortfalls and input surpluses is determined rather than merely a single efficiency score. The DEA approach secures a Pareto-Koopmans efficient function from the very nature of the DEA models, which, as shown in Foundations, represent the Charnes-Cooper test for Pareto-Koopmans optimality (efficiency).

We can illuminate other differences between DEA and these other approaches by distinguishing between "pure predictions" and "control predictions." As is noted on pp. 101 ff. in Charnes, Cooper, Phillips and Learner (1985), the usual least squares-regression approach is pointed toward pure prediction in the sense of its ability to extrapolate or interpolate the behavior of the dependent variable (i.e., the regressant) without attempting to alter underlying patterns of behavior exhibited by past data. DEA, on the other hand, provides formulas for projecting inefficient DMUs onto the efficiency frontier as in Charnes, Cooper and Rhodes (1978). The presumption is that the observed behavior for these adjusted DMUs can be altered to conform with these projections. In contrast to this "control" approach, the analyses development in Diewert and Parkan (1983) is restricted to measures of efficiency (efficiency scores) for passive use and, in addition, the failure to allow for the presence of non-zero slack renders these measures unsuitable for use in effecting such projections.

Modifications of the CCR production possibility sets, hence, the DEA model employed, may be needed of course, when some of the indicated projections cannot be attained because some components in the input vectors are partially or wholly non-discretionary. Charnes, Cooper, Rousseau and Semple (1987) provide the necessary modifications to the model to conform to managerially possible production for both cases. A treatment had been suggested in Banker and Morey (1986) for providing modified efficiency scores and thereby efficient DMUS when inputs and outputs can be identified into two classes according to whether their values are (or are not) completely fixed exogenously. It has not been shown, however, that their modified test correctly yields Pareto-Koopmans efficiency or that a correct production possibility set is effectively achieved. No such developments are to be found in the traditions of the literature covered by Hanoch and Rothschild and Diewert and Parkan which may be thought of as being more oriented toward "understanding" rather than the "understanding for use" point of view described in Charnes, Cooper, Learner and Phillips (1985).

The tradition represented by Hanoch and Rothschild and Diewert and Parkan is not the only one that can be identified in the literature of econometrics. Starting with the classic article by Farrell (1957)[1], another related series of efforts in econometrics is directed toward developing scalar overall as well as component measures of technical, scale and allocative efficiencies. Recent articles in this tradition may be found in Färe and Lovell (1978) as well as in Russell (1985) with a comprehensive and detailed treatment being available in Färe, Grosskopf and Lovell (1985).

Starting with the "CCR ratio form" as given in Charnes, Cooper and Rhodes (1978) the DEA literature has emphasized explicitly developed methods for locating sources and estimating amounts of inefficiency directly from the data. This has continued into subsequent formulations which embody the DEA principle in different forms (and different production possibility sets) such as the "invariant multiplicative form" of Charnes, Cooper, Seiford and Stutz (1983) and the "additive model" of Charnes, Cooper, Golany, Seiford and Stutz (1985) as well as the "extended additive model" as

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[1] See also the earlier article by Debreu (1951).

given in Charnes, Cooper, Rousseau and Semple (1987). In the following we restrict ourselves to the first three, the necessary elaborations for the latter being in progress.

3 DEA Geometry and Efficient Function Construction

The CCR ratio model may be rendered as the dual (non-Archimedean) linear programs:

$$
\text{(CCR)}\qquad
\begin{array}{ll}
\text{(R)} & \\
\max h = \mu^T y_0 & \\
\qquad \nu^T x_0 = 1 & \\
\qquad \mu^T Y - \nu^T X \leq 0 & \\
\qquad -\mu^T \leq -\epsilon e^T & \\
\qquad -\nu^T \leq -\epsilon e^T &
\end{array}
\qquad\qquad
\begin{array}{l}
\text{(DEA)} \\
\min \theta - \epsilon e^T s^+ - \epsilon e^T s^- \\
Y\lambda - s^+ = y_0 \\
\theta x_0 - X\lambda - s^- = 0 \\
\\
\lambda, s^+, s^- \geq 0
\end{array}
$$

where $Y \triangleq [y_1, \ldots, y_n]$, $X \triangleq [x_1, \ldots, x_n]$ are matrices of positive vectors and ϵ is a non-Archimedean infinitesimal.

The additive model is

$$
\text{(A)}\qquad
\begin{array}{lll}
\min & -e^T s^+ - e^T s^- & \\
& Y\lambda - s^+ & = y_0 \\
& -X\lambda \qquad - s^- & = -x_0 \\
& e^T\lambda & = 1 \\
& \lambda, s^+, s^- \geq 0 &
\end{array}
$$

If one wishes this value to be independent of the units of measurement we alter the functional to

$$
\text{(A.2)}\qquad e^T D^{-1}(y_0)s^+ + e^T D^{-1}(x_0)s^- = d^w(s^+, s^-)
$$

where $D(y_0)$, $D(x_0)$ are diagonal matrices with the y_0 or x_0 component entries. (If some components of y_0 or x_0 are zero we use the unique Moore-Penrose generalized inv ~se which has zeros instead of reciprocals for the zero components.) For either functional we obtain the same DMUS as efficient since

$$\text{(A.2.1)} \qquad \alpha(e^T s^+ + e^T s^-) \leq e^T D^{-1}(y_0) s^+ + e^T D^{-1}(x_0) s^- \leq \beta(e^T s^+ + e^T s^-)$$

where α, β are respectively the minimum and maximum of the non-zero entries of both $D^{-1}(y_0)$ and $D^{-1}(x_0)$ and therefore $\alpha, \beta > 0$. It is useful to have the (A.1) functional form both for theory and practice since the efficiency hyperplane it defines $(e^T s^+ + e^T s^- = 0)$ does not depend on any particular x_0, y_0.

The Invariant Multiplicative model is formally the same as the Additive model but with input and output components replaced by their logarithms.

As may be noted (cf. Foundations), the DEA side for each corresponds to a Charnes-Cooper test for Pareto-Koopmans efficiency over the relevant production possibility set. Formally, the test could be applied to arbitrary combinations of input-output vector pairs which have no correspondence to actual production function possibilities let alone efficient ones. Thus in all our DEA applications care has been taken to choose inputs and outputs with the expectation that for each single output increases in inputs should not cause decrease in the output. As shown in Foundations, this is a property of Pareto-Koopmans efficient empirical functions with a single output and generalizations to multiple output situations are also given there. The efficient multiple output and input functions need not be isotone, although for any pair of observed inputs there is a cone of directions in output space for which isotonicity holds in these ouput directions.

The DEA tests associate with each DMU tested efficient DMUs whose convex (conical for the CCR model) combinations of input-output vectors form a "facet" of efficient input-output vectors. Each facet corresponds to a piece of the efficient empirical function.

To generate the efficient DMUs associated with a particular DMU we need first the following new lemma on optimal solutions to the general linear programming problem (with an arbitrary ordered field of scalars)

$$\begin{aligned} \max \quad & c^T\lambda \\ & P\lambda = P_0 \\ & \lambda \geq 0 \end{aligned}$$

This problem may be rewritten in terms of a basic solution $\bar{\lambda}^T = (\bar{\lambda}_B{}^T, 0)$ and associated decompositions $c^T = (c_B{}^T, c_N{}^T)$, $P = [B, N]$, $\lambda^T = (\lambda_B{}^T, \lambda_N{}^T)$ as

$$\begin{aligned} \max \quad & c_B{}^T\lambda_B + c_N{}^T\lambda_N \\ & B\lambda_B + N\lambda_N = P_0 \\ & \lambda_B, \lambda_N \geq 0 \end{aligned}$$

The lemma may now be stated:

<u>Lemma</u>: If $\lambda_j > 0$ in some optimal solution (basic or not)

then in every <u>optimal</u> basic solution tableau its reduced cost (shadow price) is zero.

<u>Proof</u>: For every basis B, the constraint equation can be solved to yield

$$\lambda_B = B^{\#}P_0 - B^{\#}N\lambda_N$$

where $B^{\#}$ is any left inverse of B. Thus

$$c_B{}^T\lambda_B + c_N{}^T\lambda_N = c_B{}^TB^{\#}P_0 - (c_B{}^TB^{\#}N - c_N{}^T)\,\lambda_N$$

If B now corresponds to an optimal basic solution $\bar{\lambda}^T = (\bar{\lambda}_B{}^T, 0)$, we have

$$c^T\bar{\lambda}^T = c_B{}^T\bar{\lambda}_B = c_B{}^T B^{\#} P_0$$

as the optimal value.

Thus

$$c_B{}^T B^{\#} P_0 = c^T B^{\#} P_0 - (c_B{}^T B^{\#} N - c_N{}^T)\, \lambda_N$$

for any optimal solution $\lambda^T = (\lambda_B{}^T, \lambda_N{}^T)$ and thereby

$$(c_B{}^T B^{\#} N - c_N{}^T)\, \lambda_N = 0$$

But in an optimal basic solution tableau, the reduced cost vector $(c_B{}^T B^{\#} N - c_N{}^T) \geq 0$

Since also $\lambda_N \geq 0$, we have a sum of non-negatives equal to zero in this equation and thereby

$$(c_B{}^T B^{\#} N_j - c_j{}^T)\, \lambda_j = 0, \qquad \text{for each } j \in N$$

Thus $\lambda_j > 0$ implies $c_B{}^T B^{\#} N_j - c_j{}^T = 0.$

Q.E.D

As an illustration consider

$$\max \quad x_3 \text{ subject to } 0 \leq x_i \leq 1, \qquad i = 1, 2, 3$$

Introducing slacks to convert the inequalities to equations, an optimal tableau is

	$\vec{c}$	0	0	0	1	0	0	0
c_B	B	P_0	x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	1	1			1		
0	s_2	1		1			1	
1	x_3	1			1			1
	$\vec{\zeta}$	1	0	0	0	0	0	1

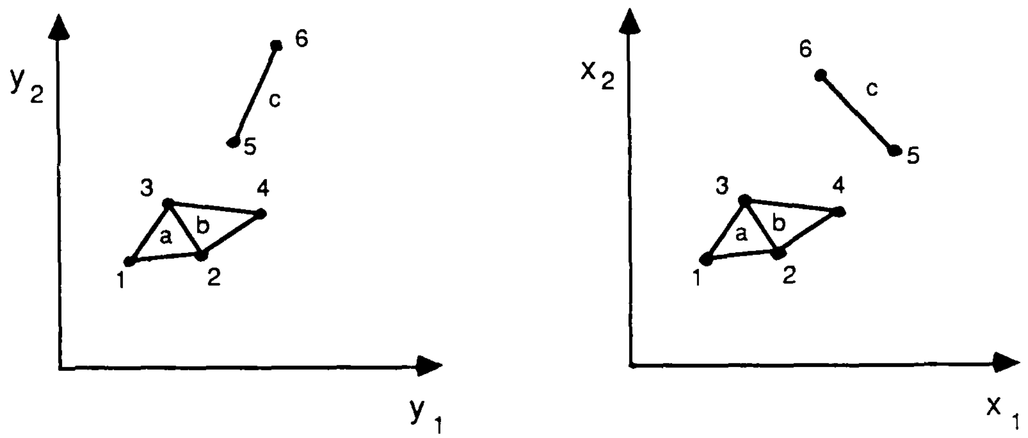
Note that x_1, x_2, x_3, s_1, s_2, may all occur in optimal solutions whereas s_3 cannot e.g. (0,0,1,1,1,0), (1,0,1,0,1,0), (0,1,1,1,0,0), (1,1,1,0,0,0). Further note (1,1,1,0,0,0) is <u>not</u> an adjacent extreme point to the optimal basic solution whose tableau we have used.

Since DMUs whose reduced cost is zero in an optimal basic solution (to the C^2 test) correspond to alternate basic optimal solutions, they are efficient as well as those in the basic optimal solution (cf. Foundations). Thus we have

<u>Theorem 1</u>: The DMUs appearing in an optimal basic solution to a DEA test together with those whose reduced costs are zero are the efficient generators of the facet corresponding to the DMU being evaluated.

To illustrate empirical efficient function development, consider the following 2 input, 2 output graphs for the additive model:

<u>Figure 1</u>



Here 6 efficient DMUs input-output points are plotted in separate input and output spaces rather than the input-output product space of the C^2 test. We suppose they are the generators of the facets a, b, c corresponding to 3 inefficient DMUs not shown. Note that so far as the empirical efficient production function is concerned we have it defined only over the input domain labeled by a, b, c, a geometrical situation which often occurs with empirical distribution functions in statistics.

Now we utilize some of our experiences in actual uses of DEA to remark that in every application we have encountered since the beginning of DEA, the following properties were present. Every facet's generators has consisted of DMUs whose

(a) input vectors are linearly independent

(b) number does not exceed the number of inputs

Theorem 2: If the DMU observation set has properties (a) and (b) then

(i) the efficient empirical production function is uniquely defined over input simplexes corresponding to each facet, i.e., for any input point which is a convex combination of the facet generators its output point is the same convex combination of the generator outputs,

(ii) it is linear on these facets

and,

(iii) continuous when crossing the boundaries between two facets.

Proof: Because of properties (a) and (b) the convex hull of the facet generators in the product input-output space defines a set of input-output points which are efficient and for which any input point in the convex hull of the inputs has a unique representation in input space in terms of the input vectors of the facet generators since by (a) and (b) this convex hull is a simplex in input space. The value of the function in the output space at this input point is the output vector which is the same convex combination of the output vectors of the facet generators. Thus this function is linear over this input vector simplex.

The boundary between two adjacent facets consists of convex combinations of the subset of input-output points which are in both facets. The value of the production function at a point on the boundary is thus obtained from the same (unique) combination of the DMUs generating these boundaries and hence its value is the same for both adjacent facets. Therefore, the empirical production function is continuous across the boundary between adjacent facets.

Q.E.D.

Thus each linear piece of the function is easily specified analytically without further computation and a path is open for further extensions (and uses) of DEA. These extensions can move from considerations of efficiency to considerations of effectiveness and studies of stability or senstivity can be addressed with relative ease by the mathematical programming methods that are available for these purposes. See, e.g., Charnes, Cooper, Lewin, Morey and Rousseau (1985). This work together with that of Charnes and Neralic (1986) also bears on the problems of enlargement of the empirical Pareto-Koopmans efficient production function to input vector domains which, for example, may fill in gaps between input domain pieces as shown in Figure 1. Such problems were brought forth and discussed in Foundations. Further contributions to its resolution would further illuminate possible relations between the Hanoch, Rothschild, Diewert and Parkan results and DEA contributions.

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